Closing Thur: 6.2

Closing Tues: 6.3, 6.4

Write out each formula, then plug in 1 year and simplify:

Entry Task:

Which account is best?

A: 4%, compounded semi-annually

B: 3.97%, compounded monthly

C: 3.955%, compounded continuously

The fast answer is to compute the annual percentage yield (APY) for each. Let me explain what APY is by doing the following:

A: $F = P\left(1+\frac{0.04}{2}\right)^{2t} = P\,(1.02)^{2t}$

IN ONE YEAR $\Rightarrow$ $F = P\,(1.02)^{2(1)} = P \cdot 1.0404$

MULTIPLYING BY 1.0404 $\Rightarrow$ $\boxed{\text{INCREASE OF } 4.04\%}$

B: $F = P\left(1+\frac{0.0397}{12}\right)^{12t} = P(1.003308\overline{3})^{12t}$

IN ONE YEAR $\Rightarrow$ $F = P \cdot (1.04043040038)$

MULT. BY 1.0404304 $\Rightarrow$ $\boxed{\text{INCREASE OF } 4.04304\%}$

C: $F = Pe^{0.03955t}$

IN ONE YEAR $\Rightarrow$ $F = P \cdot (1.0403425)$

MULT BY 1.0403425 $\Rightarrow$ $\boxed{\text{INCREASE OF } 4.03425\%}$

$\boxed{\text{ACCOUNT B IS BEST}}$

Discrete compounding

$$APY = \left[\left(1+\frac{r}{m}\right)^m - 1\right] \times 100\ \%$$

Continuous compounding

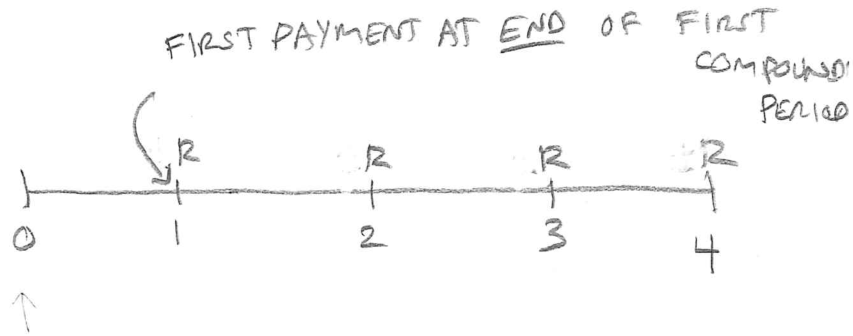
$$APY = \left[e^r - 1\right] \times 100\%$$

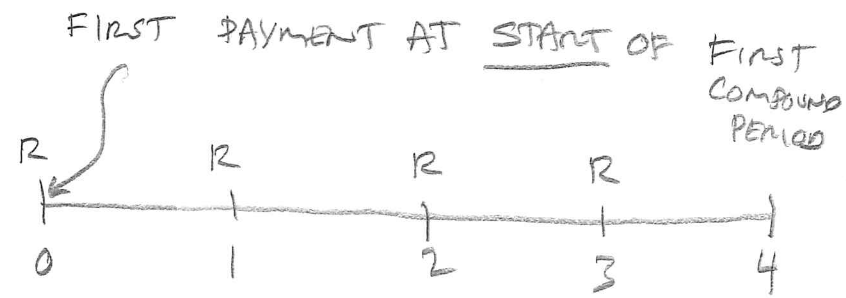
6.3 and 6.4 Annuities

An **annuity** is an interest bearing account with regular deposits or withdrawals.

Two types of Annuities:

Ordinary Annuities = payments made at END of each compounding period.





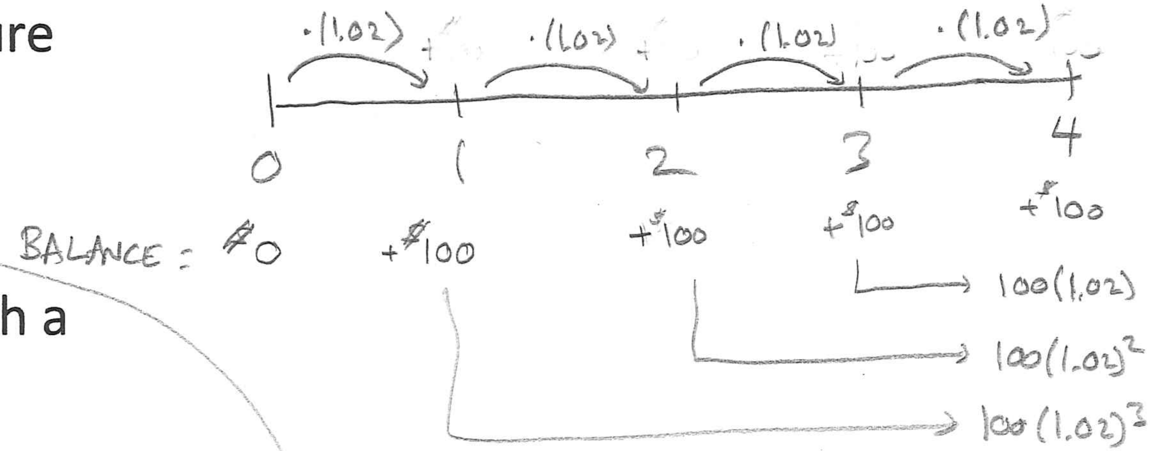
Annuities Due = payments made at the BEGINNING of each compounding period.

Two types of questions:

Future Value Questions = start with zero dollars in the account, make regular deposits, find the future value. (Balance is growing!)

INITIAL BALANCE = \$0

2% example, ORDINARY ANNUITY



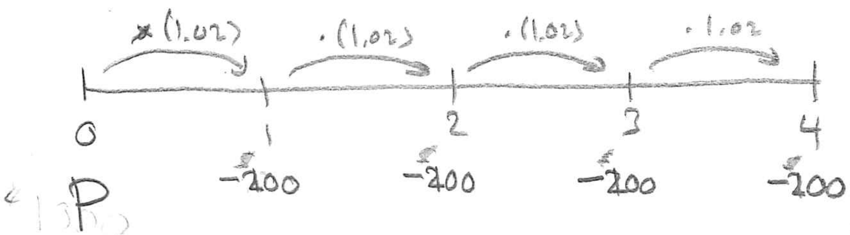
$$\text{TOTAL} = \underbrace{100 + 100(1.02) + 100(1.02)^2 + 100(1.02)^3}_{100 \cdot \frac{1.02^4 - 1}{1.02 - 1}}$$

$\approx \$409.09$ ← FUTURE BALANCE

SHORTCUT FORMULA

Present Value Questions = start with a large balance (call this *P*) in the account, make regular withdrawals, Then end with zero in the account. (Balance is shrinking!)

INITIAL BALANCE = P



BALANCE = \$0 AT END

WHAT IS P?

$$P = \frac{200}{1.02} + \frac{200}{(1.02)^2} + \frac{200}{(1.02)^3} + \frac{200}{(1.02)^4}$$

$$= 200 \cdot \left[\frac{1 - (1.02)^{-4}}{0.02}\right] \leftarrow \text{SHORTCUT FORMULA}$$

$P \approx \$761.54$

SEE POSTING!

R = amount of each regular payment

r = decimal interest rate

m = num. of compoundings in a year

ompute:

$i = \frac{r}{m}$ = rate at each compounding

$n = mt$ = total payments

$\boxed{\text{STEP 1}}$

$i = \frac{r}{m}$

$n = mt$

	Ordinary (Payments at END of each period)	**Due** (Payments at BEGINNING of each period)
FV (Balance Growing)	$F = R\frac{(1+i)^n - 1}{i}$	$F = R\frac{(1+i)^n - 1}{i}(1+i)$
PV (Balance Shrinking)	$P = R\frac{1-(1+i)^{-n}}{i}$	$P = R\frac{1-(1+i)^{-n}}{i}(1+i)$

Where do these formulas come from?

(You don't need to write this down).

First, you need to know the geometric sum.

By expanding you can see:

$$(1+x)(x-1) = x^2 - 1$$
$$(1+x+x^2)(x-1) = x^3 - 1$$
$$(1+x+x^2+x^3)(x-1) = x^4 - 1$$

and so on ...

In each case, dividing by x – 1 gives

$$1 + x = \frac{x^2 - 1}{x - 1}$$
$$1 + x + x^2 = \frac{x^3 - 1}{x - 1}$$
$$1 + x + x^2 + x^3 = \frac{x^4 - 1}{x - 1}$$

Thus, in general,

$$1 + x + x^2 + \cdots + x^{n-1} = \frac{x^n - 1}{x - 1}$$

For example:

$$1 + (1.02) + \cdots + (1.02)^7 = \frac{(1.02)^8 - 1}{1.02 - 1}$$

Second, consider an annuity with regular payments at the end of each quarter for 2 years and 8%, compounded quarterly.

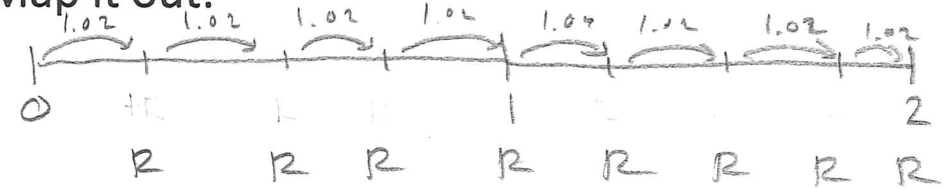
t = 2 years

r = 0.08, m = 4,

i = r/m = 0.02 (rate used each quarter)

n = mt = 8 payments

Map it out:

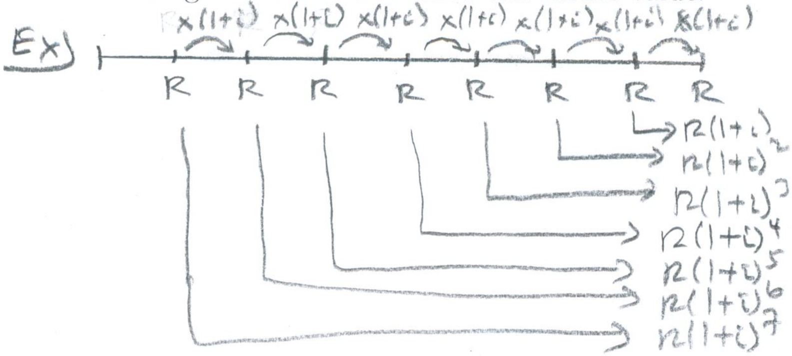


$$F = R(1.02)^7 + R(1.02)^6 + \cdots + R(1.02) + R$$
$$= R\left[(1.02)^7 + (1.02)^6 + \cdots + (1.02) + 1\right]$$

$$i = \frac{r}{m}$$

The four scenarios: $n = mt$

1. **Ordinary Annuity Future Value**: Payments at the END of each compounding period. You are given or want to know the future value.



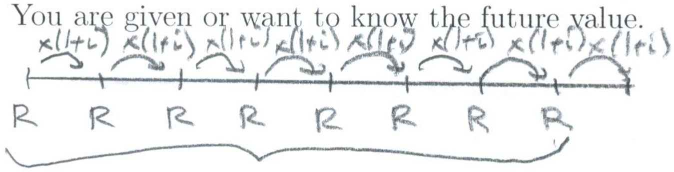
$$\text{TOTAL} = R\left[1+(1+i)+(1+i)^2+\cdots+(1+i)^7\right]$$

$$= R\,\frac{(1+i)^8 - 1}{(1+i) - 1}$$

$$= R\,\frac{(1+i)^8 - 1}{i}$$

$$\boxed{F = R\left[\frac{(1+i)^n - 1}{i}\right]}$$

2. **Annuity Due Future Value**: Payments at the BEGINNING of each compounding period. You are given or want to know the future value.

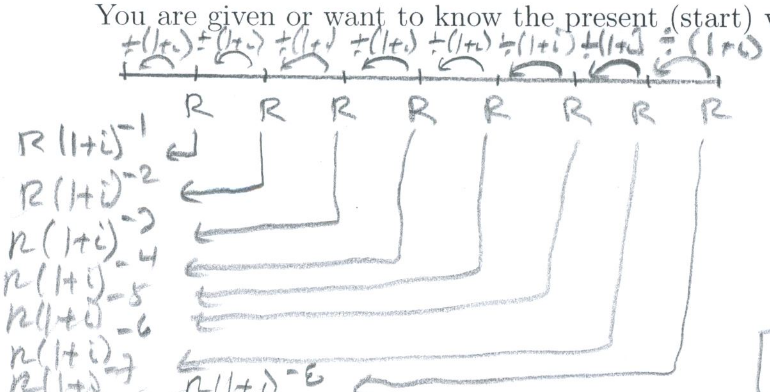


$$\text{TOTAL} = R\,\frac{(1+i)^8 - 1}{i}\,(1+i)$$

SAME AS ABOVE
BUT WITH ONE
EXTRA COMPOUNDING
PERIOD AT THE END

$$\boxed{F = R\left[\frac{(1+i)^n - 1}{i}\right](1+i)}$$

3. **Ordinary Annuity Present Value**: Payments at the END of each compounding period. You are given or want to know the present (start) value.



$$\text{TOTAL} = R\left[\frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \cdots + \frac{1}{(1+i)^8}\right]$$

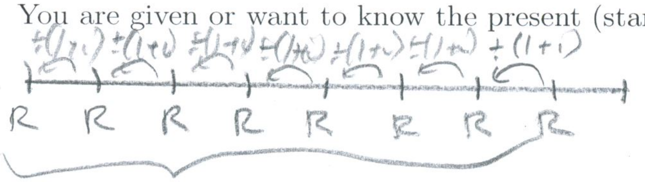
$$= \frac{R}{(1+i)}\left(1 + (1+i)^{-1} + \left((1+i)^{-1}\right)^2 + \cdots + \left((1+i)^{-1}\right)^7\right)$$

$$= \frac{R}{(1+i)}\,\frac{\left((1+i)^{-1}\right)^8 - 1}{(1+i)^{-1} - 1}$$

$$= R\,\frac{(1+i)^{-8} - 1}{1 - (1+i)} = R\,\frac{1 - (1+i)^{-8}}{i}$$

$$\boxed{P = R\left[\frac{1 - (1+i)^{-n}}{i}\right]}$$

4. **Annuity Due Present Value**: Payments at the BEGINNING of each compounding period. You are given or want to know the present (start) value.



$$\text{TOTAL} = R\left(\frac{1 - (1+i)^{-8}}{i}\right)(1+i)$$

SAME AS ABOVE
BUT WITH ONE
LESS DIVISION

$$\boxed{P = R\left[\frac{1 - (1+i)^{-n}}{i}\right](1+i)}$$

FROM THE LECTURE PACK:

1. At the <u>end</u> of each month, you place \$100 into an account bearing 6% interest, compounded monthly. What is the balance of the account 5 years after you start?

Ordinary or Due? , FV or PV?

$r = 0.06,\ m = 12 \Rightarrow i = \frac{r}{m} = \frac{0.06}{12} = 0.005$

$t = 5 \quad \Rightarrow \quad n = 5 \cdot 12 = 60$ months

$F = \overset{100}{R} \cdot \frac{(1.005)^{60} - 1}{0.005}$

$\approx \boxed{\$6977.00}$

TOTAL PAYMENTS $= R \cdot n = \$100 \cdot 60 = \$6000$

INTEREST $= \$977$

2. A company establishes a sinking fund to pay a debt of \$100,000 due in 4 years. At the <u>beginning</u> of each six-month period, they deposit \$R in an account paying 9%, compounded semi-annually. How big must the payments be to pay the debt on time?

NEED \$100,000 BALANCE IN 4 YEARS

Ordinary or Due? , FV or PV?

$r = 0.09,\ m = 2 \Rightarrow i = \frac{r}{m} = 0.045$

$t = 4 \quad \Rightarrow \quad n = mt = 8$ PAYMENTS

$\overset{\$100{,}000}{F} = R \cdot \underbrace{\frac{(1.045)^{8} - 1}{0.045} \cdot (1.045)}$

$100000 = R \cdot 8.1637125\overline{3}$

$R = \frac{100000}{8.1637125\overline{3}} \approx \boxed{\$12249.33}$

TOTAL PAYMENTS $= R \cdot n = R \cdot 8 \approx 97{,}984.^{64}$

INTEREST $\approx \$2005.^{36}$

3. Your retirement account earns 7%, compounded quarterly. How much must the account contain when you retire if you want to withdraw $6000 at the end of each quarter for 30 years?

Ordinary or Due? , FV or PV?

$r=0.07,\ m=4 \Rightarrow i=\frac{r}{m}=0.0175$

$t=30 \Rightarrow n=mt=120$ quarters

$P=R\,\frac{1-(1.0175)^{-120}}{0.0175}$ (R = \$6000)

$P \approx \boxed{\$300102.52}$

TOTAL PAYMENTS $= R\cdot n = 6000\cdot 120$
$= \$720{,}000$

INTEREST $= \$419897.48$

4. You inherit $200,000 and invest it at 3%, compounded monthly. If you withdraw $1000 at the beginning of every month, how long will the money last?

Ordinary or Due? , FV or PV? ($200,000)

$r=0.03,\ m=12 \Rightarrow i=\frac{r}{m}=0.0025$

$t=?$ $\Rightarrow n=12t$

$P=R\,\frac{1-(1.0025)^{-n}}{0.0025}\cdot 1.0025$ (P = 200000, R = 1000)

$200000=1000\,\frac{(1-(1.0025)^{-n})}{0.0025}\cdot 1.0025$

$0.49875312=1-(1.0025)^{-n}$

$(1.0025)^{-n}=0.501246883$

$-n\ln(1.0025)=\ln(0.501246883)$

$n \approx 276.6077923$ MONTHS

$12t \approx 276.6077923$

$\boxed{t \approx 23.0506 \text{ years}}$

Several annuity examples from your financial lifetime

A Student Loan: Let's say you accumulate \$40,000 in student loans and you have graduated so it is now time to repay the loan. Most student loans are paid over 10 years and the typical rate is around 4.5%, compounded monthly. What will your monthly payments be? (Assume the first payment is at the end of the first month after graduation)

Answer: First, note that $i = \frac{r}{m} = \frac{0.045}{12} = 0.00375$ and there will be $n = mt = 12 \cdot 10 = 120$ payments. This is an ordinary annuity, present value question (as loans typically are). The balance starts at \$40,000 and we will pay it down to zero. So

$$P = R\frac{1-(1+i)^{-n}}{i} \quad \text{becomes} \quad 40,000 = R\frac{1-(1.00375)^{-120}}{0.00375} \quad \text{which becomes} \quad 40,000 = R \cdot 96.489323986$$

Dividing gives

$$R = \frac{40000}{96.489323986} \approx 414.553635 = \$414.55 \text{ monthly payment for 10 years }.$$

Other notes: In total you will make 120 payments of \$414.55, so
Total Payments $= R \cdot n = 414.55 \cdot 120 = \$49,746$, which means you will pay
Total Interest $= R \cdot n - P =$ \$9,746.

Saving for retirement: Your company opens a 401(k) retirement account for you. Today (at the beginning of the first month), you got your first paycheck and it says there will be \$500 removed from your paycheck each month and placed in your 401(k) retirement. You plan to retire in 35 years and your retirement adviser conservative estimates that your 401(k) will earn approximately 6%, compounded monthly. How much will you have saved when you retire?

Answer: First, note that $i = \frac{r}{m} = \frac{0.06}{12} = 0.005$ and there will be $n = mt = 12 \cdot 35 = 420$ payments. This is an annuity due, future value question. So

$$F = R\frac{(1+i)^n - 1}{i}(1+i) \quad \text{becomes} \quad F = 500\frac{(1.005)^{420}-1}{0.005}(1.005) = 715916.92513 = \$715,916.93.$$

Other notes: In total you will make 420 payments of \$500, so
Total Payments $= R \cdot n = 500 \cdot 420 = \$210,000$, which means you earn
Total Interest $= F - R \cdot n =$ \$505,916.93.

Buying a Home: You want to buy a home. You have saved enough money for a down payment of \$40,000 which your mortgage broker says allows you to get a 30-year loan that has a rate of 5%, compounded monthly. You know that you can only afford to make monthly loan payments of \$2,000. How big of a home loan can you afford (in other words, what is the most expensive home price you can afford)?

Answer: First, note that $i = \frac{r}{m} = \frac{0.05}{12} = 0.004166...$ and there will be $n = mt = 12 \cdot 30 = 360$ payments. This is an ordinary annuity (all loans are, the first payment is made at the END of the first month). And it is a present value question because you will be paying down the balance. So

$$P = R\frac{1-(1+i)^{-n}}{i} \quad \text{becomes} \quad P = 2000\frac{1-(1.004166..)^{-360}}{0.004166..} \approx 372,563.2341 = \$372,563.23$$

You can't afford a home that gives you a loan above \$372,563.23 (after your down payment).
Thus, the largest home value you can afford is \$372,563.23 + \$40,000 = \$412,563.23.

Other notes: If you get a loan of this size, then in total you will make 360 payments of \$2000, so
Total Payments $= R \cdot n = 2000 \cdot 360 = \$720,000$, which means you will pay
Total Interest $= R \cdot n - P =$ \$346,436.77.

Spending your Retirement Savings: You are 65 and you just retired. You were able to save \$1,600,000 in your retirement account which earns 5%, compounded monthly. You plan to withdraw \$8,000 at the end of every month for the rest of your life. How old will you be when the money runs out?

Answer: First, note that $i = \frac{r}{m} = \frac{0.05}{12} = 0.004166...$ and there will be $n = mt = 12t$ payments (we don't know t). This is a ordinary, present value question. So

$$P = R\frac{1-(1+i)^{-n}}{i} \quad \text{becomes} \quad 1,600,000 = 8000\frac{1-(1.004166..)^{-12t}}{0.004166..}$$

Now we solve:

$$\begin{array}{rcll}
200 & = & \frac{1-(1.004166..)^{-12t}}{0.004166..} & \text{divided by 8000} \\
0.833... & = & 1-(1.004166..)^{-12t} & \text{multiplied by 0.004166...} \\
-0.166... & = & -(1.004166..)^{-12t} & \text{subtracted 1} \\
0.166... & = & (1.004166..)^{-12t} & \text{multiplied by -1} \\
\ln(0.166...) & = & \ln((1.004166..)^{-12t}) & \text{took ln( ) of both sides} \\
\ln(0.166...) & = & -12t\ln(1.004166..) & \text{brought down power} \\
t & = & \frac{\ln(0.166...)}{-12\ln(1.004166...)} \approx 35.91 & \text{divided}
\end{array}$$

Thus, the money will be gone in 35.91 years, so you will be 65 + 35.91 = 100.91 years old (so you'll have enough money to live in this way to about an age of 101).

Other notes: There are $n = 12t = 12 \cdot 35.91 = 430.92$ or about 431 payments, so
Total Payments $\approx R \cdot n = 8000 \cdot 431 = \$3,448,000$, which means you earn about
Total Interest $= R \cdot n - P =$ \$1,848,000 (over the 36 years you have the account).